OPTIMUM VALUES OF LIFE TABLE FUNCTIONS

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1. The Problem

The object of this investigation is to isolate three important characteristics of a life table, namely;

1) the age at which the derivative of l_x, i.e., of the proportion surviving from age zero to age x (which is uniformly negative) assumes its maximum or negatively minimum value;

2) the age at which μ_x or the force of mortality $-(dl_x/dx)/l_x$ is minimum, and;

3) the age at which the expectation of life $\overset{0}{e}_x$ is maximum.

It is well known that because of relatively large value of infant mortality compared with those of early childhood ages, the derivative of l_x is large and negative at age zero. Thereafter, the derivative continues to increase till it attains a maximum at some age (usually in the age interval 10-14), and declines thereafter till l_x becomes zero. Correspondingly, the force of mortality assumes a relatively large value at age zero, begins to decrease thereafter until a minimum is reached (usually around age 12) and continues to increase and becomes quite large at the end of the age span. The pattern of variation of $\overset{0}{e}_x$ is not that apparent, and as has been shown earlier (Mitra, 1971), the maximum value of $\overset{0}{e}_x$ is reached after age zero but usually before age 5, the proof of which will be repeated in this paper for the sake of continuity.

2. Maximum value of $\overset{0}{e}_x$

By definition, $\overset{0}{e}_x = T_x/l_x$ where $T_x = \int_0^{\alpha-x} l_{x+t}dt$, α being the upper age limit, so that $l_\alpha=0$. Thus,

$$\frac{dT_x}{dx} = -l_x$$

and therefore,

$$\frac{d\overset{0}{e}_x}{dx} = -1 + \frac{T_x}{l_x^2}\left(-\frac{dl_x}{dx}\right) = \overset{0}{e}_x\,\mu_x - 1 \quad (1)$$

Now the life expectancy is known to increase after age zero, to reach a maximum before age five and to decrease thereafter. Therefore, the only optimum value of $\overset{0}{e}_x$ is the maximum which is attained at say, $x = \hat{x}$, where

$$\overset{0}{e}_{\hat{x}}\mu_{\hat{x}} - 1 = 0$$

or

$$\overset{0}{e}_{\hat{x}} = 1/\hat{\mu}_x \quad (2)$$

3. Minimum value of μ_x

For $\overset{0}{e}_x$ to be maximum at $x = \hat{x}$,

$$\frac{d^2\overset{0}{e}_x}{dx^2} = \mu_x\left(\overset{0}{e}_x\mu_x - 1\right) + \overset{0}{e}_x\frac{d}{dx}\mu_x \quad (3)$$

must be negative at $x = \hat{x}$. Because of (2),

$$\left[\frac{d^2\overset{0}{e}_x}{dx^2}\right]_{x=\hat{x}} = \overset{0}{e}_{\hat{x}}\left[\frac{d}{dx}\mu_x\right]_{x=\hat{x}} \quad (4)$$

which can be negative when

$$\left[\frac{d}{dx}\mu_x\right]_{x=\hat{x}} \text{ is negative.}$$

Again, μ_x is known to decline at age zero and to attain a minimum at, say $x = \tilde{x}$. Therefore, $d\mu_x/dx$ is negative in the age interval o to $\tilde{x}$. Clearly then, the maximum value of $\overset{0}{e}_x$ is obtained before μ_x reaches its minimum.

$$\text{Thus, } \hat{x} < \tilde{x}. \quad (5)$$

4. Maximum value of the derivative of l_x

The minimum value of $\mu_x = -(dl_x/dx)/l_x$ is reached when

$$\frac{d\mu_x}{dx} = \frac{1}{l_x^2}\left(\frac{dl_x}{dx}\right)^2 - \frac{1}{l_x}\left(\frac{d^2l_x}{dx^2}\right) \quad (6)$$

$$= \mu_x^2 - \frac{1}{l_x}\left(\frac{d^2l_x}{dx^2}\right) \quad (7)$$

is zero at $x = \tilde{x}$, so that

$$l_{\tilde{x}}\mu_{\tilde{x}}^2 = \left[\frac{d^2l_x}{dx^2}\right]_{x=\tilde{x}} \quad (8)$$

Now, the derivative of l_x is uniformly negative, and because of large force of mortality at age zero, assumes a maximum value at, say $x = \bar{x}$. Correspondingly, the second derivative of l_x is positive at age zero, decreases thereafter till it becomes zero at $x = \bar{x}$.

Since the right hand side of (8) is positive, it is clear that the minimum value of the force of mortality is reached before the derivative of l_x assumes its maximum or its smallest negative value. In other words,

$$\tilde{x} < \bar{x} \tag{9}$$

and combining (9) with (5), the following inequality relationship,

$$\hat{x} < \tilde{x} < \bar{x} \tag{10}$$

is established.

5. Estimation of $\bar{x}$, $\tilde{x}$ and $\hat{x}$

The exact age at which the three optimum conditions are met cannot be obtained since none of the life table functions l_x, μ_x, or $\overset{0}{e}_x$ can be expressed in terms of easily differentiable mathematical functions. Approximate solutions were however, obtained earlier (Mitra, ibid.) for ages corresponding to maximum life expectancies for different life tables. The method used was to find the point of intersection of two freehand curves obtained by plotting the values of $\overset{0}{e}_x$ and $1/\mu_x$ at early childhood ages. The values of μ_x can be determined from probabilities of dying ${}_nq_x$ in the age interval x to x+n, because of the well-known relationship

$$1-{}_nq_x = e^{-\int_0^n \mu_{x+t}\,dt} \tag{11}$$

The exponent $\int_0^n \mu_{x+t}\,dt$ can be approximated by $n\mu_{x+\frac{n}{2}}$ $(n<5)$

for the age interval 1 to 20 where the force of mortality is relatively small and finally, disregarding squares and higher powers of μ_x, (11) can be reduced to

$${}_nq_x = n\mu_{x+\frac{n}{2}} \tag{12}$$

Accordingly, the age, say x', at which ${}_nq_x$ is minimum, is approximately related to $\tilde{x}$ by the following equation,

$$\tilde{x} = x' + n/2 \tag{13}$$

The method of finite differences can then be used to determine x' for which ${}_nq_x$ values for n=5 and x=5, 10 and 15 are sufficient, since x' can generally be located around age ten. Using the conventional notation Δ for successive differences, so that

$$\Delta({}_nq_x) = {}_nq_{x+n} - {}_nq_x$$

and

$$\Delta^2({}_nq_x) = \Delta({}_nq_{x+n}) - \Delta({}_nq_x)$$

etc., the solution for $\tilde{x}$ is given by

$$\tilde{x} = 10 - \frac{5\Delta({}_5q_5)}{\Delta^2({}_5q_5)} \tag{14}$$

disregarding differences of third and higher orders.*

6. Applications

The formulas developed in the preceding sections for the estimation of $\hat{x}$, $\tilde{x}$ and $\bar{x}$, and their inequality relationships were tried on a few model life tables (United Nations, 1958) for males, selected to cover a wide range of expectations of life at birth. The results are shown in Table 1.

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*According to Newton's forward difference formula, the function u_x, when u_0, u_1, u_2,...are known can be expressed as $u_x = u_0 + x\Delta u_0 + \frac{x(x-1)}{1.2}\Delta^2 u_0 + \ldots\ldots$ The function u_x can then be differentiated to determine optimum values, point of inflection, etc.

Table 1. Optimum values and respective ages of a few life table functions

Model life Table number	Expectation of life at birth $\overset{0}{e}_o$	Age of maximum life expectancy $\hat{x}$	Maximum life expectancy $\overset{0}{e}_{\hat{x}}$	Age of minimum force of mortality $\tilde{x}$	Age of maximum dl_x/dx $\overline{x}$
(1)	(2)	(3)	(4)	(5)	(6)
10	24.8	4.5	38.1	13.2	13.7
25	31.9	3.6	43.8	12.8	13.1
40	39.2	2.8	49.5	12.6	12.8
55	46.4	2.6	55.6	12.3	12.5
70	53.6	2.0	60.5	12.1	12.2
85	61.5	1.7	65.0	11.7	11.8
100	68.5	0.9	69.9	11.1	11.2

The sharp increase in life expectancy from age 0 to $\hat{x}$ is worth noting. So is the declining trend of $\hat{x}$ with increase in $\overset{0}{e}_o$. For life tables with lower life expectancies, the age of maximum value of $\overset{0}{e}_x$ is larger and this is due to high early childhood mortality in addition to high infant mortality. Both $\tilde{x}$ and $\overline{x}$ are large compared to $\hat{x}$, and their variations over the wide range of life tables are systematic and small. Like $\hat{x}$, $\tilde{x}$ and $\overline{x}$ are also inversely related with life expectancy, and cover a range of 13.2 to 11.1 and 13.7 to 11.2 respectively over the range of $\overset{0}{e}_o$ used in this analysis.

As noted in (9) $\tilde{x} < \overline{x}$, but the difference is rather small in most cases.

Summary

The characteristics of life table functions are quite well known but investigations of some of the crucial values of these functions and their interrelationships have not yet been carried out. The importance of such an inquiry need not be over-emphasized in view of the fact that any exercise on graduation of life table functions must take into account the order and spacing of these parametric values. It is known that the function l_x has a steep negative slope at age zero and the steepness continues to diminish till it becomes smallest at some age and begins to increase thereafter.

Similarly, the force of mortality μ_x assumes a minimum value at some age and life expectancy $\overset{0}{e}_x$ attains a maximum value after age zero. These three ages have been found to form a sequence with the age corresponding to the maximum life expectancy as the lowest of the series. Methods have been outlined to estimate these values for a given life table and all three are found to be inversely related with expectation of life at birth.

References

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